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Optimization in curbing risk contagion among financial institutes☆

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ABSTRACT

Financial institutions are interconnected by holding debt claims against each other. A default bank may cause its creditors to default, and the risk may be further propagated to up-stream institutes (risk contagion). Such interconnection is a key contributing factor to the past worldwide financial crisis. We show that a good mechanism of default liquidation may improve the total wealth of the financial system and therefore may curb the risk contagion. We formulate this problem as a nonlinear optimization problem with constraints and propose an optimal liquidation policy to minimize the system's loss. We show that the problem resembles a Markov decision problem (MDP) and therefore we can apply the direct-comparison based optimization approach to solve this problem. Higher order directional derivatives and some optimality properties are obtained. Furthermore, we derive an iterative algorithm which combines both the policy iteration and the gradient based approach to find a local optimal policy, and under some conditions, a global optimal policy. Our work provides a new direction in curbing the risk contagion in financial networks; and it illustrates the advantages of the direct-comparison based approach, originated in the field of discrete event dynamic system, in nonlinear optimization problems.

1. Introduction

This paper is motivated by two recent developments in financial engineering and performance optimization. First, financial institutions are interconnected by borrowing-and-lending activities among themselves or holding marketable securities against each other (Chen, Liu, & Yao, 2014). Such interconnection is a critical influencing factor to the past worldwide financial crisis and the European sovereignty debt crisis, and could potentially threaten the stability of financial networks. For example, a default bank may cause its creditors to default, and the risk may be further propagated to up-stream institutes. During default liquidation, it is extremely significant to curb such risk contagion among financial networks. The goal of this paper is to propose a new liquidation scheme based on performance optimization and to provide computation algorithms that solve the problem.

The problem formulated above has a large dimension and many highly nonlinear constraints, and we need to develop an efficient algorithm for an optimal solution. On this side, a direct-comparison based approach has been developed in the past years to the optimization of nonlinear problems and has been successfully applied to many problems, such as optimization of singular controlled diffusion processes (Ni & Fang, 2013), MDP with long-run average criterion (Cao, 2007, 2015), and variance criterion (Xia, 2016), and nonlinear performance with probability distortion (Cao & Wan, 2017). In this paper, we show that the special features of the financial risk contagion problem make it possible to be solved by the direct-comparison based approach, leading to some new insights to the problem.

While network connections have a positive effect by diversifying risk (Haldane, 2009; Summer, 2013), it can have a negative effect by adding spreading channels for risks. When a global shortage for liquidity happens, the systemic risk will be transmitted through the risk-sharing mechanism (Allen & Gale, 2000; Leitner, 2005; Rochet & Tirole, 1996). Such risk contagions may result in consecutive consequences for the financial system, such as declines in asset prices, higher price volatility, more bank defaults, and market inefficiency (Allen & Gale, 2004; Brunnermeier & Pedersen, 2009; Holmstrom & Tirole, 2000).

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In response to the financial crisis and its consequences, it is natural to expect the central bank (CB) to take an active role in controlling systemic risk. Yellen's speech appeals governments to improve financial stability and reduce the risks posed by the interconnected financial system (Yellen, 2013). Dasgupta (2004) claims that a CB is indispensable no matter the structure of a financial system is complete or not. Castiglionesi (2007) presents a model in which a CB can prevent financial contagion by imposing reserve requirements.

In this paper, we propose another possible role that the CB may take in curbing contagion: arbitrating the liquidation among banks in the system during financial crisis, and providing required compensation to achieve fairness. In the literature, it is often assumed that all financial institutes are paid off in proportion to the size of their claims on bank assets (Chen et al., 2014; Eisenberg & Noe, 2001). We show that by allowing different liquidation schemes we may reduce the system's total debts and save banks from defaulting. However, this may violate the fairness of the prorated scheme, and CB arbitration and compensation are needed; this is consistent with both Yellen's speech (Yellen, 2013) and others' work on the roles of CBs (Castiglionesi, 2007; Dasgupta, 2004).

The above problem can be formulated as a performance optimization problem. However, it has a large dimension and many highly nonlinear constraints and is therefore difficult to analyze. Fortunately, we solve this problem through an approach originally developed for the optimization of discrete event dynamic systems (Cao, 2007), called the direct-comparison based approach. The approach is intuitively clear, and it can provide new insights, leading to new results to many problems (Cao, 2007, 2015; Cao & Wan, 2017; Ni & Fang, 2013; Xia, 2016).

More precisely, a liquidation scheme determines how the financial institutes pay their debts among themselves, and is also called a policy. After implementing a liquidation scheme, some banks are default and some others non-default, leading to a partition of default and non-default banks. All the policies leading to the same partition are called a region in the policy space. The policy space may have multiple regions corresponding to different partitions. The optimization problem for policies in the same region was reported in the conference paper (Ye, Xue, Gao, & Cao, 2017). In this paper, we study both the one-region and multi-region optimization problems; we show that by choosing the right liquidation scheme, we may significantly reduce system's total debts and save banks from defaulting.

The difficulty of this optimization problem is mainly caused by the nonconvexity of the regions and nonlinear constraints involved; and there are discontinuities in the performance gradients across the boundaries of the different regions in the policy space. In this paper, we apply the direct comparison based approach to solve this optimization problem in curbing risk contagion among financial institutes. The approach is based on a performance difference formula (PDF), and this PDF explicitly shows the cost difference across different regions. Based on it, we may develop a policy iteration–performance gradients combined algorithm, which leads to a local optimal policy, and under some conditions, a global optimal policy. Numerical examples indicate a significant improvement in the system's loss and the number of default banks. Our work casts new insights to the problem, extends the Eisenberg–Noe model (Eisenberg & Noe, 2001) and obtains the optimal liquidation scheme in curbing risk contagion.

The reminder of the paper is organized as follows. In Section 2, we review the Eisenberg–Noe model (Eisenberg & Noe, 2001) and other related works, and we formulate the optimization problem. In Section 3, we apply the direct-comparison based approach and propose a policy iteration–gradient combined algorithm for the optimal liquidation scheme to minimize the possible system's loss. In Section 4, we provide two numerical examples to demonstrate the efficiency of our approach. Finally, we conclude this paper in Section 5.

2. Problem formulation

2.1. A brief review

Our work is based on the structural framework for contagion in financial network proposed in Eisenberg and Noe (2001). This model illustrates how shocks to individual agents can be propagated through interbank networks, and it was followed by many subsequent works (Chen et al., 2014; Glasserman & Young, 2015; Liu & Staum, 2010).

There are n banks with interconnected balance sheets. The banks in the financial system may have liabilities to each other. The interconnection of the banks is represented via an $n \times n$ liability matrix $L := (L_{i,j})$, where $L_{i,j}$ denotes the nominal obligation of bank i to bank j. Naturally, $L_{i,j} \geq 0$ for $i \neq j$ and $L_{i,i} = 0$. Every bank may also have some liabilities to creditors outside the network, denoted as a row vector $b = (b_i)$, $b_i \geq 0$. We denote the liability vector as $l := (l_i)$, $l_i := b_i + \sum_{j \neq i} L_{i,j}$, and assume $l_i > 0$, for $i = 1, 2, \ldots, n$. We set $r_{i,j} := L_{i,j}/l_i$ to denote the relative liability, and let $R := (r_{i,j})$. We assume that there is only one seniority for the liability. We use a vector $\alpha := (\alpha_i)$, $\alpha_i \geq 0$, to represent the values of exogenous assets of the banks. Then the total asset of bank i is $\alpha_i + \sum_{j \neq i} L_{j,i}$. A bank is defined to be in default if its total liability exceeds its total asset. It is often assumed that bank default will not change the prices outside the network, i.e., α is independent of defaults.

Let $P := (p_{i,j})$ be the liquidation matrix of a liquidation scheme, meaning that in this scheme bank i pays bank j proportionally to $p_{i,j}, j \neq i, j = 1, 2, \ldots, n$. More precisely, let x_i be the total debt that bank i pays to others, then bank i pays bank j with $x_i p_{i,j}$. $x = (x_i)$ is called a clearing vector. In normal situation, $P = R$. It is called a pro rata scheme, i.e., debts are paid proportionally to the relative liabilities.

Next, x and P satisfy the following conditions:

a. *Limited liability*.

$$x_i \leq \alpha_i + \sum_{j=1}^{n} x_j p_{j,i}, \quad \forall i = 1, 2, \ldots, n.$$

b. *Absolute priority*. Either liabilities are paid in full $x_i = l_i$, or all value is paid to creditors, i.e.,

$$x_i = \alpha_i + \sum_{j=1}^{n} x_j p_{j,i}, \quad \forall i = 1, 2, \ldots, n.$$

Putting them into a matrix form as the fixed-point characterization, we have

$$x = \min[l, \alpha + xP]. \tag{1}$$

Based on fixed-point arguments, Eisenberg and Noe (2001) proves that a clearing vector exists for any realization of α, and under some mild regularity conditions, the clearing vector is unique. As an example, if every bank has a positive external liability, i.e., $b_i > 0$ for all i (Glasserman & Young, 2015), then the matrix P is substochastic (all the row sums are strictly less than 1), and thus there exists a unique clearing vector x, because the above fixed-point formulation (1) is a contraction.

Simple and fast algorithms have been developed to calculate the clearing vector x. Eisenberg and Noe (2001) shows that it can be obtained by solving the following linear programming problem, with the performance measure $\eta := \sum_{i=1}^{n} x_i$:

$$\max_{x} \ \eta, \quad s.t., \quad x(I - P) \leq \alpha, \quad 0 \leq x \leq l. \tag{2}$$

As an improvement of the linear programming approach, a partition algorithm has been developed (Chen et al., 2014; Staum, Feng, & Liu, 2016). Let $D = \{i : x_i < l_i\}$ and $N = \{i : x_i = l_i\}$ be the default set and the non-default set. Partition P so that

$$P = \begin{pmatrix} P_D & P_{D,N} \\ P_{N,D} & P_N \end{pmatrix},$$

where P_D and P_N correspond to sets D and N, respectively. Then the problem (2) is equivalent to solving the following optimization:

$$\begin{aligned} &\max_{x_D, D, N} \quad \eta \\ \text{s.t.} \quad & x_D = \alpha_D + x_N P_{N,D} + x_D P_D, \; x_N = l_N, \quad (3) \\ & x_N \leq \alpha_N + x_N P_N + x_D P_{D,N}, \; x_D < l_D. \quad (4) \end{aligned}$$

P_D is substochastic and thus $(I_D - P_D)^{-1}$ exists (Chen et al., 2014). Then (3) is equivalent to

$$x_D = (\alpha_D + l_N P_{N,D})(I_D - P_D)^{-1}, \; x_N = l_N. \quad (5)$$

The Partition Algorithm determines the default set D and the non-default set N resulted by a liquidation scheme P, and x can then be easily determined by (5). In this paper, all the P's corresponding to the same partition D and N are called a region in the policy space of P. A bank d with $l_d = \alpha_d + \sum_{i=1}^{n} x_i p_{i,d}$ can be either in D or N; such a bank is called a *boundary bank*.

Another related work is the sensitivity analysis of the Eisenberg–Noe model of contagion by Liu and Staum (2010). Define the terminal wealth v of the banks as $v = \alpha + xP - x = \alpha + x(I - P)$. Liu & Staum (2010) derives the derivatives of the payment x and wealth v w.r.t. the exogenous (initial) wealth α, $\frac{\partial x}{\partial \alpha}$ and $\frac{\partial v}{\partial \alpha}$. In particular, it shows that at a boundary bank, the left and right-sided derivatives are different, i.e., $\frac{\partial^+ x}{\partial \alpha} \neq \frac{\partial^- x}{\partial \alpha}$ and $\frac{\partial^+ v}{\partial \alpha} \neq \frac{\partial^- v}{\partial \alpha}$.

2.2. Our model

Most existing works deal with the pro rata liquidation scheme with $P = R$. It naturally raises the following question: can we adjust the scheme P to reduce the system's loss and the number of default banks? If so, what is the best liquidation scheme? We first consider a simple example.

Example 1. There are 4 banks in the system with liabilities $L_{1,2} = L_{2,3} = L_{3,4} = L_{4,3} = 2$, $L_{1,4} = 8$, and $L_{i,j} = 0$ for all other terms. The exogenous assets are $\alpha_1 = 5$, $\alpha_j = 0, j \neq 1$; and the outside debts are $b_3 = 2, b_j = 0, j \neq 3$. Following the pro rata scheme with $P = R$, $r_{1,2} = 0.2$, $r_{1,4} = 0.8$, $r_{2,3} = r_{4,3} = 1$, $r_{3,4} = 0.5$, and $r_{i,j} = 0$ for all other terms, we have $x = (5, 1, 3, 2)$ and $\eta = 11$. There are three default banks, $D = \{1, 2, 3\}$ and $N = \{4\}$. However, if we slightly adjust the scheme, e.g., apply $p_{1,2} = 0.4$, $p_{1,4} = 0.6$, and $p_{i,j} = r_{i,j}$ for all other terms instead, then we have $x' = (5, 2, 4, 2)$ and $\eta' = 13$ (We denote the values under the new scheme by a prime in the superscript). It can be verified that it has only one default bank, $D = \{1\}$ and $N = \{2, 3, 4\}$. □

This example shows that if we apply a different liquidation scheme, we may indeed reduce the system's total debts and save some banks from defaulting. The price we pay is that "fairness" in the normal sense may be violated. However, this may provide a possible way to curb risk contagion for the CB to consider. The sacrificed fairness may be properly compensated in some way, e.g., by setting future liabilities between both sides of banks who benefited and suffered from the scheme, or by offering tax benefits to those who suffer, etc.

Motivated by the example, we state our problem as the following optimization problem:

$$\max_{P} \quad \{\max_{x} \quad \eta\} \quad (6)$$

$$\begin{aligned} \text{s.t.} \quad & x = \min[l, \alpha + xP], \quad (7) \\ & p_{i,i} = 0, p_{i,j} \geq 0, \sum_j p_{i,j} = 1 - b_i/l_i, \\ & \forall i, j = 1, 2, \ldots, n, i \neq j. \end{aligned}$$

The performance measure to be optimized in this problem is $\eta = \sum_{i=1}^{n} x_i$, as suggested in Cont, Moussa, and Santos (2010) and Eisenberg and Noe (2001). As shown in (2), the "$\max_x$" in (6) is simply for determining the value of clearing vector according to a fixed P; and the control variable is the matrix P. We refer to a liquidation scheme P as a policy in the terminology of optimization. η is all the payments made by the whole system. If there is no bank default, $\eta = \sum_{i=1}^{n} l_i$; otherwise $\eta < \sum_{i=1}^{n} l_i$. Maximizing η is equivalent to minimizing the total loss of the financial system $\sum_{i=1}^{n} l_i - \eta$.

Because x depends on P via (7), the variable x can be considered as an implicit function of P. Then the problem is a nonlinear bilevel (leader–follower) problem, which contains two levels of variables. However, the gradients $\nabla_P[\eta(P, x)]$ and $\nabla_P[x(P)]$ are not continuous because of the nonconvex feasible region. Compared with existing works on bilevel programming (Colson, Marcotte, & Savard, 2005, 2007; Kolstad & Lasdon, 1990), the above problem has a large dimension, highly nonlinear constraints and discontinuous gradients. Moreover, the policy space may have multiple regions, which brings challenges for determining the effect on η when the partition changes. Fortunately, this problem can be solved by the direct-comparison based approach.

In this paper, we want to find an optimal scheme P_{max} among all regions. While under some conditions, there is only one region in the policy space. Then the problem becomes a one-region optimization problem as reported in the conference paper (Ye et al., 2017). Here, we give a sufficient condition under which there is only one region in the policy space:

$$\begin{cases} \alpha_i + \sum_{j \neq i} (l_j - b_j) < l_i, & \text{for } i \in D, \\ \alpha_i \geq l_i, & \text{for } i \in N. \end{cases}$$

In the remainder of the paper, we apply the direct-comparison based approach; its central piece is a performance difference formula (PDF) that provides all the details of the difference of the performance under any two policies, including those in different regions. Based on the PDF, we develop an efficient algorithm for the optimal liquidation scheme, which combines policy iteration and gradient together.

3. Optimal liquidation scheme

3.1. Performance difference formula

We first derive the PDF. To this end, let P and P' denote two different policies; the quantities related to P' are denoted by a prime in the superscript, such as x'. According to (3), we have

$$(x_D \quad x_N) = (x_D \quad x_N) \begin{pmatrix} P_D & 0 \\ P_{N,D} & 0 \end{pmatrix} + (\alpha_D \quad l_N).$$

Rewrite it as

$$x = x\bar{P} + \beta,$$

where

$$\bar{P} = \begin{pmatrix} P_D & 0 \\ P_{N,D} & 0 \end{pmatrix}, \; \beta = (\alpha_D \quad l_N).$$

Then we have

$$x = \beta(I - \bar{P})^{-1},$$
$$\Pi := (I - \bar{P})^{-1} = \begin{pmatrix} (I_D - P_D)^{-1} & 0 \\ P_{N,D}(I_D - P_D)^{-1} & I_N \end{pmatrix}.$$

In the default set D, we define the depth of nodes $\rho_D = (I_D - P_D)^{-1}e = (I_D + P_D + P_D{}^2 + \cdots)e$, where $e = (1, 1, \ldots, 1)^T$ is a column vector with all components being one, and it has a proper dimension making the related matrix operation meaningful. ρ_D measures the amplification of losses due to interconnections among nodes in the default set (Glasserman & Young, 2015). Similarly, we can define ρ as follows:

$$\rho = (I - \bar{P})^{-1}e = \begin{pmatrix} \rho_D \\ P_{N,D}\rho_D + e_N \end{pmatrix}, \tag{8}$$

where $e_N = (1, 1, \ldots, 1)^T$ is a column vector with all components being one, and it has the same dimension with x_N. Then we have

$\eta := xe = \beta\rho.$

Thus $(I - \bar{P})\rho = e$, $x'e = x'(I - \bar{P})\rho$, and

$x'(\bar{P}' - \bar{P})\rho = x'(I - \bar{P})\rho + x'(\bar{P}' - I)\rho = x'e - \beta'\rho.$

Then we can derive the PDF as follows:

$$x'e - xe = x'(\bar{P}' - \bar{P})\rho + (\beta' - \beta)\rho, \tag{9}$$

with the constraints (3) and (4).

In the PDF (9), β, β', $\bar{P}$, and $\bar{P}'$ depend on the partitions D, N and D', N'. Next, we derive a more explicit PDF indicating the effect of the change of the scheme from P to P'. Define $\tilde{N}$ as the set of all banks moving from D to N', and $\tilde{D}$ as the set of all banks moving from N to D'. Then (9) takes the following form:

$$\begin{aligned} x'e - xe = & \sum_{i=1}^{n}[\sum_{j\in D} x'_i(p'_{i,j} - p_{i,j})\rho_j] \\ & + \sum_{j\in\tilde{N}}[(l_j - \alpha_j) - \sum_{i=1}^{n} x'_i p'_{i,j}]\rho_j \\ & + \sum_{j\in\tilde{D}}[\sum_{i=1}^{n} x'_i p'_{i,j} + (\alpha_j - l_j)]\rho_j \\ = & A_1 + A_2 + A_3, \end{aligned} \tag{10}$$

where A_1, A_2, and A_3 denote the terms in the three lines on the right-hand side of the above equation, respectively. Because the spectral radius of P_D is less than 1 (Chen et al., 2014), we know $\rho_j > 0$, for $j = 1, 2, \ldots, n$. If bank $j \in N$, then $l_j \le \alpha_j + \sum_{i=1}^{n} x_i p_{i,j}$, and if $j \in D$, then $l_j \ge \alpha_j + \sum_{i=1}^{n} x_i p_{i,j}$. The following lemma specifies the values of A_2 and A_3.

Lemma 1. *In (10), it holds that $A_2 \le 0$ and $A_3 \le 0$.*

Proof. If $j \in \tilde{N}$ from D to N', which means under the new scheme P', bank j will be non-default. Thus, we have $l_j \le \alpha_j + \sum_{i=1}^{n} x'_i p'_{i,j}$, then

$$[(l_j - \alpha_j) - \sum_{i=1}^{n} x'_i p'_{i,j}]\rho_j \le 0,$$

where "=" holds only when $\tilde{N} = \emptyset$, or contains only boundary points.

For $j \in \tilde{D}$ from N to D', which means under the new scheme P', bank j will be default. Thus, we have $l_j \ge \alpha_j + \sum_{i=1}^{n} x'_i p'_{i,j}$, then

$$[\sum_{i=1}^{n} x'_i p'_{i,j} + (\alpha_j - l_j)]\rho_j \le 0,$$

where "=" holds only when $\tilde{D} = \emptyset$, or contains only boundary points. □

The intuitive reason for this lemma is that when bank d changes from D to N', the maximum payout for bank d becomes l_d, and the payment it receives is $\alpha_d + \sum_{i=1}^{n} x'_i p'_{i,d}$. Considering the amplification factor ρ_d, the amount of $[\alpha_d + \sum_{i=1}^{n} x'_i p'_{i,d} - l_d]\rho_d$ is wasted; on the other hand, when bank $d+1$ changes from N to D', the payout for bank $d+1$ was calculated as l_{d+1}, however, after the change its payout in fact is only $\sum_{i=1}^{n} x'_i p'_{i,d+1} + \alpha_{d+1}$. Considering the amplification factor ρ_{d+1}, the total payment after the change should be reduced by the amount of $[l_{d+1} - (\sum_{i=1}^{n} x'_i p'_{i,d+1} + \alpha_{d+1})]\rho_{d+1}$. Compared with one-region optimization (Ye et al., 2017), the PDF (10) explicitly shows the cost across partitions from D, N to D', N', through the last two terms A_2 and A_3, respectively.

Surprisingly, the PDF (9) resembles the PDF for a Markov decision problem with P as the transition probability matrix and x as the states of a Markov system (Cao, 2007); although there is no dynamics in the current problem. Since the number of banks is much smaller than the number of states in a Markov process, the dimension of the risk contagion problem is much smaller than an MDP. Because of Lemma 1, policy iteration cannot be directly applied in different regions; we need to use the performance gradients as well. In the next section, we derive the directional derivative of η with respect to policy changes.

3.2. Directional derivatives

For any $\delta \in [0, 1]$, P and P', let $P(\delta) = P + \delta Q = P + \delta(P' - P)$ and $\eta(\delta) := x(\delta)e$ be the corresponding performance, with $x(0) = x$ and $x(1) = x'$. The directional derivative along the direction defined by $Q = P' - P$ is (cf. Cao (2007))

$$\frac{d\eta(\delta)}{d\delta}|_{\delta=0} = \lim_{\delta\downarrow 0}\frac{\eta(\delta) - \eta}{\delta}.$$

If P is not a boundary point, then for any P', when δ is small enough, $P(\delta)$ and P are in the same region. Similar to the matrix P, we can partition Q accordingly as

$$Q = \begin{pmatrix} Q_D & Q_{D,N} \\ Q_{N,D} & Q_N \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} Q_D & 0 \\ Q_{N,D} & 0 \end{pmatrix}.$$

Then we have

$$x(\delta)e - xe = \sum_{i=1}^{n}[\sum_{j\in D} x_i(\delta)(p'_{i,j} - p_{i,j})\rho_j]\delta,$$

and

$$\frac{d\eta(\delta)}{d\delta}|_{\delta=0} = \sum_{i=1}^{n}[\sum_{j\in D} x_i(p'_{i,j} - p_{i,j})\rho_j] = x\bar{Q}\rho. \tag{11}$$

This derivative can also be obtained by taking derivatives of $x(\delta)[I - \bar{P}(\delta)] = \beta$, with $\bar{P}(\delta) = \bar{P} + \delta\bar{Q}$, i.e.,

$$\frac{dx(\delta)}{d\delta} = x(\delta)\bar{Q}[I - \bar{P}(\delta)]^{-1}.$$

Taking derivatives on both sides with respect to δ yields

$$\begin{aligned} \frac{d^2x(\delta)}{d\delta^2} &= \frac{dx(\delta)}{d\delta}\bar{Q}[I - \bar{P}(\delta)]^{-1} + x(\delta)\bar{Q}\frac{d}{d\delta}[I - \bar{P}(\delta)]^{-1} \\ &= 2x(\delta)\bar{Q}[I - \bar{P}(\delta)]^{-1}\bar{Q}[I - \bar{P}(\delta)]^{-1}, \end{aligned}$$

and

$$\frac{d^2x(\delta)}{d\delta^2}|_{\delta=0} = 2x\bar{Q}[I - \bar{P}]^{-1}\bar{Q}[I - \bar{P}]^{-1}.$$

In general, we have the nth directional derivatives

$$\frac{d^nx(\delta)}{d\delta^n}|_{\delta=0} = n!x\{\bar{Q}[I - \bar{P}]^{-1}\}^n.$$

At boundary points, different Q's may point to different regions. Since a boundary point can be viewed as in any neighboring regions, the above derivation still holds for directional derivatives; and the set D in (11) may be different for different directions.

3.3. The optimization algorithm

Based on the PDF (10) and Lemma 1, we propose the following optimization algorithm to obtain the optimal scheme P_{max}.

Algorithm 1. 1. Choose the initial liquidation matrix as the relative liability matrix, i.e., set $P_0 := R$, and set $k = 0$.
2. Determine D, N for P_k by the Partition Algorithm, or by solving the linear programming (2), in Section 2.1. Calculate η_k.
3. Calculate ρ for P_k by (8), denote $P_k = P$, and determine a P' by

$$\max_{p'_{i\bullet}} \sum_{j\in D}(p'_{i,j} - p_{i,j})\rho_j, \quad i = 1, 2, \ldots, n, \tag{12}$$

where $p'_{i\bullet}$ denotes the ith row of the matrix P'. The choice may not be unique. If $p_{i\bullet}$ reaches the maximum, choose $p'_{i\bullet} := p_{i\bullet}$.
4. If $p'_{i\bullet} := p_{i\bullet}$, for all $i = 1, 2, \ldots, n$, i.e.,

$$\arg\{\max_{p'_{i\bullet}}[\sum_{j\in D}(p'_{i,j} - p_{i,j})\rho_j]\} = p_{i\bullet}, \tag{13}$$

for all $i = 1, 2, \ldots, n$, then stop. Otherwise, go to Step 5.
5. Set $Q = P' - P$ and $P(\delta) = P + \delta Q$.
6. Find a δ^* such that $\eta(\delta^*)$ is as large as possible in the direction. If $\eta(\delta^*) < \eta$, then reduce the size of δ^* until $\eta(\delta^*) > \eta$; avoid boundary points in the process.
Set $P_{k+1} = P(\delta^*)$ and $k := k + 1$, then go to Step 2. □

If all the policies are in the same region, then the A_2 and A_3 in (10) are zero, and Algorithm 1 becomes a pure policy iteration algorithm. This is discussed in Ye et al. (2017). In this case, all the policies have the same number of default banks, but the algorithm helps reducing the total debts among all banks. If the algorithm does not stop in Step 4, then the directional derivative along Q is positive and therefore $\eta(\delta^*) > \eta$ if δ^* is small enough. In each iteration we go along the direction with the largest performance derivative, and this is the same as policy iteration in Markov decision processes (MDPs) (Cao, 2007). Compared with the one-region optimization problem discussed in Ye et al. (2017), we extend the search space from one region to multiple regions; and therefore, the performance can be further improved and some banks in danger of default can be saved. However, because the policies may go across different regions, it is not necessary that the performance still improves after moving over the boundary; therefore we need to adjust the step size as in Step 6.

3.4. Some implementation issues

In Step 6 of Algorithm 1, boundary points are avoided; in real implementation, because of the discrete nature of sampling, this is naturally implemented. However, if the optimal policy is on the boundary, because of the continuity of η in P, the algorithm may end up with an approximation that may as close as we wish to the true optimal. This is common to numerical implementations.

We found another issue when implementing Algorithm 1. As the performance improves in each step, the performance sequence η_k, $k = 1, 2, \ldots$, converges. However, following the deterministic sequence generated by (12), the convergence rate may be very slow, or it may even converge to a value less than the maximum. To avoid this, we need to add some randomness in Step 5. We modify it as

Step 5′: Choose $1 > \theta > 0$, With probability $1 - \theta$, set $Q = P' - P$. With probability θ, set $Q = P'' - P$, where P'' is any other matrix such that

$$\sum_{j\in D}(p''_{i,j} - p_{i,j})\rho_j > 0, \quad i = 1, 2, \ldots, n.$$

Set $P(\delta) = P + \delta Q$. □

The performance derivative along P to P'' is also positive; the iteration can be carried on. We call the algorithm with Step 5 replaced by Step 5′ the **Modified Algorithm.**

The last implementation issue is the stopping criteria. In practice, Algorithm 1 may go on forever. This issue can be resolved in a standard way: choose a predetermined integer $K > 0$ and a small positive number $\epsilon > 0$, and stop the iteration process if in K consecutive iterations the improvement in performance is less than ϵ.

3.5. Properties of the algorithm

We say that a policy P reaches a local maximum if the directional derivative along any direction is non-positive. A policy P reaches a global maximum if no other policy has a better performance. By (1), $l_i \geq x_i$ for all $i = 1, 2, \ldots, n$. In addition, we assume that $x_i > 0$ for all $i = 1, 2, \ldots, n$; i.e., every bank with debts pays something in any scheme. We can obtain the following Lemma 2.

Lemma 2. *If P is not a boundary point and its performance η is a local maximum, it is also a global maximum.*

Proof. Let P with D and N be a local maximum. Suppose the lemma is not true, then there is a P', with D' and N', and x', such that $x'e > xe$. We use the PDF (10). By Lemma 1, $A_2 \leq 0$ and $A_3 \leq 0$. Thus, there must be a bank, denoted as i^*, such that $\sum_{j\in D} x'_{i^*}(p'_{i^*,j} - p_{i^*,j})\rho_j > 0$. Because $x'_{i^*} > 0$, we have $\sum_{j\in D}(p'_{i^*,j} - p_{i^*,j})\rho_j > 0$. Next, we construct a policy P^* with

$$p^*_{i,j} = \begin{cases} p'_{i^*,j}, & \text{for } i = i^*, j = 1, 2, \ldots, n, \\ p_{i,j}, & \text{for } i \neq i^*, j = 1, 2, \ldots, n. \end{cases}$$

Set $P(\delta) = P + \delta Q^*$; we can choose δ small enough so that $P(\delta)$ and P are in the same region (i.e., with the same D and N). Then the directional derivative along $Q^* = P^* - P$ from P is

$$\frac{d\eta(\delta)}{d\delta} = \sum_{j\in D} x^*_{i^*}(p'_{i^*,j} - p_{i^*,j})\rho_j > 0,$$

where $x^*_{i^*}$ corresponds to P^*. This contradicts to the fact that P is a local maximum. □

The lemma may not hold if P is a boundary point, because the P^* constructed in the lemma and P may not be in the same region.

Lemma 3. *If Algorithm 1 stops in Step 4, it stops at a local maximum as well as a global maximum.*

Proof. According to (13), if the algorithm stops at $\hat{P}$, then the directional derivative along any direction at $\hat{P}$ is non-positive and $\hat{P}$ is a local maximum. The lemma then follows from Lemma 2 directly. □

Lemma 4. *Let P and P' be the two policies in Step 3 of Algorithm 1, and η and η' be the corresponding performance measures, respectively. If $\eta' - \eta < 0$, P' and P are in two different regions.*

Proof. In each iteration, according to (12), we have $\sum_{j\in D}(p'_{i,j} - p_{i,j})\rho_j \geq 0$, $i = 1, 2, \ldots, n$. Thus, we have $A_1 \geq 0$ in the PDF. Because $\eta' < \eta$, we have $A_2 + A_3 < 0$ by Lemma 1. Then P' nd P are in different regions. □

Both Algorithm 1 and the Modified Algorithm provide a sequence of P_k's with increasing η_k's. By Lemma 3, if it stops, it stops at an optimal policy. However, when the optimal policy is a boundary point, or in some rare cases, Algorithm 1 may never stop. But we have

Lemma 5. *The sequence η_k, $k = 1, 2, \ldots$, generated by the Modified Algorithm converges to the maximum value η_{max}, with probability one.*

Proof. We have shown that $\eta_{k+1} > \eta_k$ if the algorithm does not stop at the kth iteration. Then the sequence η_k, $k = 1, 2, \ldots$, must converge to some value $\hat{\eta}$, we prove that $\hat{\eta} = \eta_{max}$. Suppose this is not true. Let $V := \{P \in \mathcal{R}^{n\times n},\ \eta \geq \hat{\eta}\}$, where η is the performance corresponding to P. V is a compact set in $\mathcal{R}^{n\times n}$. Now, because $\eta_k \to \hat{\eta}$, by continuity, P_k can be as close as possible to the boundary of V. By the randomness in Step 5′, the Modified Algorithm will eventually choose a policy pointing to the inside of V with probability one, resulting in a policy with performance larger than $\hat{\eta}$. That means it is not possible (with probability zero) to keep all $\eta_k < \hat{\eta}$ forever in the iteration process, for any $\hat{\eta} < \eta_{max}$. The lemma thus holds. □

While there is a trade-off between performance and iteration times of the algorithm. Algorithm 1 may converge to a local optimal policy with fewer iterations. While the Modified Algorithm can converge to the global optimal policy with probability 1, with more iterations. Therefore, one may choose Algorithm 1 to improve the performance given fixed small number of iterations. Also, randomness adds some additional cost for exploration. Thus, sometimes one may wish to save it, so to choose Algorithm 1. On the other hand, if one wants to improve the performance further, he/she can choose the Modified Algorithm at the cost of possible more iterations. Therefore, we list these two algorithms for people to choose according to different requirements.

4. Examples

We illustrate our method with some examples.

Example 2. There are 3 banks in the system. All banks are interconnected by holding debt claims against each other. This relationship can be presented by the liability matrix L. The exogenous assets and outside liabilities of banks can be denoted as vectors α and b, respectively. The parameters are as follows:

$$L = \begin{pmatrix} 0 & 40 & 40 \\ 20 & 0 & 60 \\ 5 & 5 & 0 \end{pmatrix},\ \alpha = (41, 42, 50),\ b = (0, 10, 10).$$

Therefore, the pro rata scheme matrix is

$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.22 & 0 & 0.67 \\ 0.25 & 0.25 & 0 \end{pmatrix}.$$

The total liability of the system is $\sum_{i=1}^{3} l_i = 190$. While by the Partition Algorithm, the original payment is $x = (63.5, 78.75, 20)$, and the total payment is $\eta = 162.25$. That means, with this scheme, the system loses some money and there are two default banks, 1 and 2, i.e., $D = \{1, 2\}$, $N = \{3\}$.

We choose the pro rata scheme P as an initial policy and apply the Algorithm 1. The algorithm stops at Step 4 through only 1 iteration. As Lemma 2 states, Algorithm 1 converges to a local as well as global maximum. We get the optimal payment $x^* = (80, 90, 20)$ and the total payment $\eta^* = 190$. That means all the banks are safe (non-default) and the new scheme is

$$P_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 0.89 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}.$$

We can also apply the Modified Algorithm. The algorithm converges to the same global maximum, i.e., the optimal payment $x^* = (80, 90, 20)$ and the total payment $\eta^* = 190$, but through 2 iterations and with a different scheme

$$P_M^* = \begin{pmatrix} 0 & 0.634 & 0.366 \\ 0.383 & 0 & 0.507 \\ 0.227 & 0.273 & 0 \end{pmatrix}.$$

In this case, we can find that compared with the Modified Algorithm, Algorithm 1 can achieve the global optimal policy within fewer iterations.

Example 3. There are 30 banks in the system with the liability matrix L and vectors α and b. The data of this system are generated randomly. For the limit of space, we omit the detailed data of this example. Because of heavy liquidity, the whole system is confronted with a severe systemic crisis. With the pro rata scheme, all banks in the system will default except Bank 19, with a total payment $\eta = 68034.47$.

Without randomness, Algorithm 1 converges to a local maximum with a significant improvement of performance. By setting a stopping criterion, Algorithm 1 yields a new liquidation scheme P_1, with a total payment $\eta_1 = 75181.23$. This scheme curbs the financial contagion by reducing 8.13% of the system's loss. This is achieved only by improving the payment scheme, and no additional money is needed from the CB or government.

Then we apply the Modified Algorithm. The algorithm yields an alternative liquidation scheme, with a total payment $\eta^* = 132583.13$. This is much larger than both the results of Algorithm 1 and the pro rata scheme. This scheme curbs the financial contagion well, totally reducing 73.45% of the system's loss. In addition to Bank 19, there are totally 20 banks safe now. Therefore, the Modified Algorithm is demonstrated to be more efficient for reducing the system's loss and the number of default banks.

5. Conclusion

In this paper, we study the risk contagion problem and characterize analytically the effect of the liquidation mechanism on the total wealth of financial networks. We formulate contagion reduction as a performance optimization problem with nonlinear constraints. We apply the direct-comparison based approach to solve this problem. In this approach, we derive the performance difference formula which clearly shows the details of the differences of any two liquidation schemes, and we derive the directional derivatives of the performance measures in the policy space. Some optimality properties are obtained. Furthermore, we develop a policy iteration–gradient combined algorithm for the optimal liquidation scheme. Finally, we provide some examples to illustrate the efficiency of our algorithm for reducing the system's loss and the number of default banks.

Compared with the pro rata scheme, our proposed optimal liquidation scheme reduces the system's total debts and save banks from defaulting. This provides a new direction for curbing the contagion among financial institutes for the government and central bank to consider during financial crisis.

The direct-comparison based approach to performance optimization was first developed for discrete event dynamic systems, and has been applied to many theoretical as well as practical problems. In this paper, we find that surprisingly the performance difference formula for the risk contagion problem looks similar to that in MDPs. Our research indicates that this approach can also be applied to static problems such as the risk contagion problem. In addition, since for some other problems, "fairness" is not an essence, our results of this paper can be extended to these contexts, such as power grids systems, logistics systems, and some other network systems.

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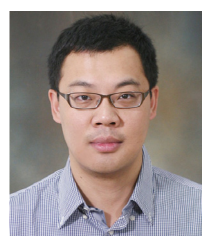

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