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An Iterative Approach to Reduce Systemic Risk among Financial Institutes ⋆

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Abstract: Financial institutions are interconnected by holding debt claims against each other. The interconnection is a key contributing factor to the past worldwide financial crisis. A default bank may cause its creditors to default, and the risk may be further propagated to up-stream institutes. We study how the mechanism of default liquidation affects the total wealth of the financial system and curbs the risk contagion. We formulate this problem as a nonlinear optimization problem with equilibrium constraints and propose an optimal liquidation policy to minimize the system's loss without changing the partition of default and non-default banks. We show that the optimization problem resembles a Markov decision problem (MDP) and therefore we can apply the direct-comparison based optimization approach to solve this problem. We derive an iteration algorithm which combines both the policy iteration and the gradient based approach. Our work provides a new direction in curbing the risk contagion in financial networks; and it illustrates the advantages of the direct-comparison based approach, which originated in the field of discrete event dynamic system, in nonlinear optimization problems.

Keywords: Direct-comparison based optimization, discrete event systems, financial network, Markov decision problems, risk contagion, policy iteration

1. INTRODUCTION

The paper is motivated by two recent developments in financial engineering and performance optimization. First, it is known that financial institutions link with each other through borrowing-and-lending activities among themselves or holding marketable securities against each other (Chen et al. [2014]). Since the 2007-2009 credit crisis in the US and the European sovereignty debt crisis, it has been increasingly clear that the interconnection could pose potential threats to the stability of the financial system. For example, a default bank may cause its creditors to default, and the risk may be further propagated to upstream institutes. During default liquidation, how to curb such risk contagion among financial networks becomes extremely important. In this paper, We show that this issue can be modeled as an optimization problem.

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The problem formulated above has a large dimension and many highly nonlinear constraints, and we need to develop an efficient algorithm for an optimal solution. On this side, a direct-comparison based approach has been developed in the past years to the optimization of nonlinear problems and has been successfully applied to many problems, such as optimization of singular controlled diffusion processes (Ni and Fang [2013]), MDP with long-run average criterion (Cao [2007, 2015]) and variance criterion (Xia [2016]), and nonlinear performance with probability distortion (Cao and Wan [2017]). In this paper, we show that the special features of the financial risk contagion problem make it possible to be solved by the direct-comparison based approach, leading to some new insights to the problem.

To study the rationale for the risk contagion effect, much of the literature is devoted to analyze the structure of the financial network, e.g., Catanzaro and Buchanan [2013] proposes complex networks as a tool to avert the financial crisis. It is shown that network connections can have both a positive effect by diversifying risk and a negative effect by adding spreading channels for risk (Haldane [2009], Summer [2013]). In the literature, Rochet and Tirole

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[1996], Allen and Gale [2000], and Leitner [2005], focus on how the risk-sharing mechanism can transmit the systemic crisis when a global shortage for liquidity happens. Other papers pour attention into the contagious effect of asset price, such as Holmstrom and Tirole [2000], Allen and Gale [2004], and Brunnermeier and Pedersen [2007].

According to Yellen's speech at the American Economic Association (Yellen [2013]), in response to the financial crisis and the weaknesses it revealed, governments around the globe are acting to improve financial stability and reduce the risks posed by a highly interconnected financial system. It is natural to expect the central bank (CB) to take a more active role in curbing contagion and controlling systemic risk. Dasgupta [2004] and Castiglionesi [2007] have already argued the necessity of a CB in systemic risk.

In this paper, we propose another possible role that the CB may take in curbing contagion: arbitrating the liquidation among banks in the system during the economic crisis and providing required compensation to achieve fairness. In the literature, it's often assumed that all claimant nodes are paid off in proportion to the size of their claims on bank assets , such as Eisenberg and Noe [2001] and Chen et al. [2014]. We show that by allowing different liquidation schemes we may reduce the system's loss without changing the partition of default and non-default banks. However, this may violate the fairness of the prorated scheme, and government/CB arbitration and compensation are needed; this echoes Yellen's speech (Yellen [2013]) and is consistent with others' work on the roles of governments and CBs (Dasgupta [2004], Castiglionesi [2007]).

The above problem can be modeled as a performance optimization problem. The solution space may have multi regions, i.e., different partitions of default and non-default banks. In this paper, we want to find an optimal solution in one of the regions. Fortunately, we solve this problem through an approach originally developed for the optimization of discrete event dynamic systems (Cao [2007]), called the direct-comparison based approach. The approach is based on a performance difference formula (PDF), which provides the details in the difference of the performance of the system under any two policies. The approach is intuitively clear, and it can provide new insights, leading to new results to many problems (Ni and Fang [2013], Cao [2007, 2015], Xia [2016], Cao and Wan [2017]).

In this paper, we apply this approach to the risk contagion problem and develop an algorithm that combines policy iteration with performance gradient; numerical examples indicate a big improvement in financial loss. Our work casts new insights to the problem, extends the Eisenberg-Noe model and obtains optimal liquidation scheme in curbing the risk contagion.

The rest of the paper is structured as follows. In Section 2, we review the Eisenberg-Noe model and some other related works, and we formulate our problem. In Section 3, we apply the direct-comparison based approach and propose an policy iteration-gradient combined algorithm for the optimal liquidation scheme to minimize the system's loss in one region. In Section 4, we provide two numerical examples showing the improvement in system's loss. Finally, Section 5 concludes the paper.

2. PROBLEM FORMULATION

2.1 A brief review

Our work is based on the structural framework for contagion in financial network proposed in Eisenberg and Noe [2001]. This model illustrates how shocks to individual agents can be propagated through interbank networks, and it was followed by many subsequent works, such as Liu and Staum [2010], Chen et al. [2014], and Glasserman and Young [2015].

In the model, there are n banks with interconnected balance sheets. The interconnection of the banks is represented via an $n \times n$ liability matrix $L := (L_{i,j})$, where $L_{i,j}$ denotes the nominal obligation of bank i to bank j. Naturally, $L_{i,j} \geq 0$ for $i \neq j$ and $L_{i,i} = 0$. Every bank may also have some liabilities to creditors outside the network, denoted as a row vector $b = (b_i)$, $b_i \geq 0$. (Vectors in the form of (b_i) are row vectors.) We denote the liability vector as $l := (l_i)$, $l_i := b_i + \sum_{j \neq i} L_{i,j}$, and set $r_{i,j} := L_{i,j}/l_i$ to denote the relative liability, and let $R := (r_{i,j})$. We assume that there is only one seniority for the liability. We use a vector $\alpha := (\alpha_i)$, $\alpha_i \geq 0$, to represent the values of exogenous assets of the banks. Then the total asset of bank i is $\alpha_i + \sum_{j \neq i} L_{j,i}$.

A bank is defined to be in default if its total liability exceeds its total assets. It's often assumed that bank default will not change the prices outside the network, i.e., α is independent of defaults.

Let $P := (p_{i,j})$ be the liquidation matrix, meaning that in this scheme bank i pays bank $j \neq i$ proportionally to $p_{i,j}$, $j \neq i$, $j = 1, 2, \cdots, N$. More precisely, let x_i be the total debt that bank i pays to others, and $x = (x_i)$ is called the a clearing payment vector. Then bank i pays bank j with $x_i p_{i,j}$. In normal situation, $P = R$, i.e., debts are paid proportionally to the relative liabilities, which is called a pro rata scheme. Next, x and P satisfy the following conditions:

a. *Limited Liability.* $\forall i = 1, ..., n$,

$$x_i \leq \alpha_i + \sum_{j=1}^{n} x_j p_{j,i}.$$

b. *Absolute Priority.* $\forall i = 1, ..., n$, either liabilities are paid in full $x_i = l_i$, or all value is paid to creditors, that is

$$x_i = \alpha_i + \sum_{j=1}^{n} x_j p_{j,i}.$$

Putting them into a matrix form as the fixed-point characterization, we have

$$x = \min[l, \alpha + xP]. \tag{1}$$

On the basis of fixed-point arguments, Eisenberg and Noe [2001] discusses the existence and uniqueness of the clearing vector. It is proved that, for each realization of α, a clearing vector exists. Furthermore, the clearing vector is unique under some mild regularity conditions. As an example, if we assume that every bank has positive external liability, i.e., $b_i > 0$ for all i like what Glasserman and Young [2015] does, then the substochastic matrix P has all the row sums strictly less than 1. Thus there exists a unique clearing vector x, because the above fixed-point formulation (1) is a contraction.

Simple and fast algorithms have been developed to calculate the clearing vector x. Eisenberg and Noe [2001] shows that it can be obtained by solving the following linear programming problem, with $|x| := \sum_{i=1}^{n} x_i$:

$$\max_x |x|, \quad s.t. \quad x(I-P) \le \alpha, \quad 0 \le x \le l. \qquad (2)$$

As an improvement of the linear programming approach, a partition algorithm has been developed in Chen et al. [2014] and Staum et al. [2016]. Let $D = \{i : x_i < l_i\}$ and $N = \{i : x_i = l_i\}$ be the default set and the non-default set. Partition P so that

$$P = \begin{pmatrix} P_D & P_{D,N} \\ P_{N,D} & P_N \end{pmatrix},$$

where P_D and P_N correspond to sets D and N, respectively. Then the problem (2) is equivalent to solving the following optimization:

$$\begin{aligned} &\max_{x_D, D, N} \quad |x| \\ s.t. \quad & x_D = \alpha_D + x_N P_{N,D} + x_D P_D, \qquad (3) \\ & x_N \le \alpha_N + x_N P_N + x_D P_{D,N}, \\ & x_N = l_N, \qquad (4) \\ & x_D < l_D. \end{aligned}$$

P_D is usually substochastic and thus $(I_D - P_D)^{-1}$ exists (Chen et al. [2014]). Then (3) and (4) are equivalent to:

$$x_D = (\alpha_D + l_N P_{N,D})(I_D - P_D)^{-1} \text{ and } x_N = l_N \qquad (5)$$

The partition algorithm determines the default set D and non-default set N, and x can then be easily obtained by (5). All the $P's$ corresponding to the same partition D and N are called a region in the space of P.

A bank d with $l_d = \alpha_d + \sum_{i=1}^{n} x_i p_{i,d}$ can be either in D or N; such a bank is called a *boundary bank*.

2.2 Our model

Most existing works deal with the pro rata liquidation scheme with $P = R$. It naturally raises the following questions: given a partition D, N, can we adjust the scheme P to reduce the total loss of financial system? If so, what is the best payment scheme?

In fact, if we apply a different liquidation scheme, we may indeed reduce the system's loss. The price we pay is that "fairness" in the normal sense may be violated. However, this may provide a possible way of overcoming the economic crisis for the central bank or the government to consider. The sacrificed fairness may be properly compensated in some way, for example, by setting future liabilities between both sides of banks who benefited and suffered from the scheme, or offering tax benefits to those who suffer, etc.

Therefore, we state our problem as the following optimization problem:

$$\begin{aligned} &\max_P \quad \{\max_x \quad |x|\} \qquad (6) \\ s.t. \quad & x = \min[l, \alpha + xP], \qquad (7) \\ & p_{i,i} = 0, p_{i,j} \ge 0, \sum\nolimits_j p_{i,j} = 1 - b_i/l_i, \\ & \forall i, j = 1, 2, ..., n, i \ne j. \end{aligned}$$

The performance measure to be optimized in this problem is $\eta = |x|$, as suggested in Cont et al. [2010]. As shown in (2), the "$\max_x$" in (6) is simply for determining the value of the clearing vector according to a fixed P; and the control variable is the matrix P. We refer to a liquidation scheme P as a policy in the terminology of optimization. η is all the payments made by the whole system. If there is no bank default, $\eta = \sum_{i=1}^{n} l_i$; otherwise $\eta < \sum_{i=1}^{n} l_i$. Maximizing η is equivalent to minimizing the total loss of the financial system $\sum_{i=1}^{n} l_i - \eta$.

Because x depends on P via (7), the variable x can be considered as an implicit function of P. Then the problem is a nonlinear optimization which contains two levels of variables. This problem can be transferred into a bilevel (leader-follower) problem. However, the gradients $\nabla_P \eta(P, x)$ and $\nabla_P x(P)$ are not continuous because of the nonconvex feasible region. Compared with existing works on bilevel programming (Kolstad and Lasdon [1990], Colson et al. [2005, 2007]), the above problem is more complex. More precisely, this problem has a large dimension and highly nonlinear constraints. Moreover, the solution space may have more than one region, which brings challenges for optimization. Fortunately, we solve this problem with the direct-comparison based approach.

A region of P corresponds to a partition of sets D and N. In this paper, we want to find an optimal scheme P_{max} in the same region with the pro rata liquidation scheme R. While under some conditions, there is only one region in the solution space. Here, we give a sufficient condition for this special case:

$$\begin{cases} \alpha_i + \sum_{j \ne i} (l_j - b_j) < l_i, & \text{for } i \in D, \\ \alpha_i \ge l_i, & \text{for } i \in N. \end{cases}$$

In the rest of the paper, we show that our optimization problem has the same structure as a Markov decision problem, although there is no dynamics involved; and therefore, we can apply the direct-comparison based approach (Cao [2007]); its central piece is a performance difference formula (PDF) that provides the details of the difference of the performance under any two policies. Based on the PDF, we develop an efficient algorithm which combines policy iteration and gradient together.

3. OPTIMAL LIQUIDATION SCHEME

3.1 Performance difference formula

We first derive the PDF. To this end, let P and P' denote two different policies; the quantities related to P' are denoted by a prime in the superscript, such as x', etc. According to (3) and (4), we have

$$(x_D \quad x_N) = (x_D \quad x_N) \begin{pmatrix} P_D & 0 \\ P_{N,D} & 0 \end{pmatrix} + (\alpha_D \quad l_N).$$

Rewrite it as

$$x = x\bar{P} + \beta,$$

where

$$\bar{P} = \begin{pmatrix} P_D & 0 \\ P_{N,D} & 0 \end{pmatrix}, \ \beta = (\alpha_D \quad l_N).$$

We see β only depends on the partition D and N. Then we have

$$x = \beta(I - \bar{P})^{-1},$$
$$\Pi := (I - \bar{P})^{-1} = \begin{pmatrix} (I_D - P_D)^{-1} & 0 \\ P_{N,D}(I_D - P_D)^{-1} & I_N \end{pmatrix}.$$

As for nodes in the default set D, we define the depth of nodes $\rho_D = (I - P_D)^{-1}e = (I + P_D + {P_D}^2 + \cdots)e$, where $e = (1, 1, \cdots, 1)^T$ is a column vector with all components being one, and a proper dimension making the related matrix operation meaningful. ρ_D measures the ampliation of losses due to interconnections among nodes in the default set (Glasserman and Young [2015]). Similarly, we can define ρ as follows:

$$\rho = (I - \bar{P})^{-1}e = \begin{pmatrix} \rho_D \\ P_{N,D}\rho_D + e_N \end{pmatrix}, \tag{8}$$

and

$$\eta := xe = \beta\rho.$$

Thus $(I - \bar{P})\rho = e$, $x'e = x'(I - \bar{P})\rho$ and

$$x'(\bar{P}' - \bar{P})\rho = x'(I - \bar{P})\rho + x'(\bar{P}' - I)\rho = x'e - \beta'\rho.$$

Then we can derive the PDF as follows:

$$x'e - xe = x'(\bar{P}' - \bar{P})\rho + (\beta' - \beta)\rho, \tag{9}$$

with the constraints

$$l_N \leq (x_D \quad l_N) \begin{bmatrix} P_{D,N} \\ P_N \end{bmatrix} + \alpha_N,$$

and

$$l_D > (x_D \quad l_N) \begin{bmatrix} P_D \\ P_{N,D} \end{bmatrix} + \alpha_D.$$

Because we optimize in the same region, $\beta' = \beta$, then the PDF (9) can be written as

$$x'e - xe = x'(\bar{P}' - \bar{P})\rho. \tag{10}$$

More explicitly, the PDF (10) takes the following form:

$$x'e - xe = \sum_{i=1}^{n}[\sum_{j\in D} x'_i(p'_{i,j} - p_{i,j})\rho_j] \tag{11}$$

Surprisingly, the PDF (10), or (11), looks the same as the PDF for a Markov decision problem with P as the transition probability matrix and x as the states of a Markov system (Cao [2007]); although there is no dynamics in the current problem. Since the number of banks is much smaller than the number of states in a Markov process, the dimension of the risk contagion problem is much smaller than an MDP. In the next section, we derive the directional derivative of η with respect to policy changes.

3.2 Directional derivatives

For any P and P', let $P(\delta) = P + \delta Q = P + \delta(P' - P)$ and $\eta(\delta) := x(\delta)e$ be the corresponding performance, with $x(0) = x$ and $x(1) = x'$. The directional derivative along the direction defined by $Q = P' - P$ is (cf. Cao [2007])

$$\frac{d\eta(\delta)}{d\delta}|_{\delta=0} = \lim_{\delta\downarrow 0}\frac{\eta(\delta) - \eta}{\delta}.$$

If P is not a boundary point, then for any P', when δ is small enough, $P(\delta)$ and P are in the same region. Then

$$x(\delta)e - xe = \sum_{i=1}^{n}[\sum_{j\in D} x_i(\delta)(p'_{i,j} - p_{i,j})\rho_j]\delta,$$

and

$$\frac{d\eta(\delta)}{d\delta}|_{\delta=0} = \sum_{i=1}^{n}[\sum_{j\in D} x_i(p'_{i,j} - p_{i,j})\rho_j]. \tag{12}$$

At the boundary points, different Q's may point to different regions. Because a boundary point can be viewed as in any neighboring regions, the above derivation still holds; in this paper, we always hold P in the same region.

3.3 The optimization algorithm

Based on PDF (11), we propose the following optimization algorithm to obtain the optimal scheme P_{max}.

Algorithm 1. Policy iteration and gradient based algorithm to maximize the performance η.

(Initialization)

- •1 Choose the initial liquidation matrix as the relative liability matrix, i.e., set $P_0 := R$, and set $k = 0$.

(Policy Evaluation)

- •2 Determine D, N for P_k by the Partition Algorithm, or solving the linear programming (2), in Section 2.1. Calculate η_k.

(Policy Update)

- •3 Calculate ρ for P_k by (8), denote $P_k = P$, and determine a P' by
$$\max_{p'_{i\bullet}} \sum_{j\in D}(p'_{i,j} - p_{i,j})\rho_j, \quad i = 1, 2, \cdots, n, \tag{13}$$
where $p'_{i\bullet}$ denotes the ith row of the matrix P'. The choice may not be unique. If $p_{i\bullet}$ reaches the maximum, choose $p'_{i\bullet} := p_{i\bullet}$.

(Stopping Rule)

- •4 If $p'_{i\bullet} := p_{i\bullet}$, for all $i = 1, 2, \cdots, n$, i.e.,
$$\arg\{\max_{p'_{i\bullet}}[\sum_{j\in D}(p'_{i,j} - p_{i,j})\rho_j]\} = p_{i\bullet}, \tag{14}$$
for all $i = 1, 2, \cdots, n$, then stop. Otherwise, go to the next step.
- •5 Set $Q = P' - P$ and $P(\delta) = P + \delta Q$.
- •6 Find a δ^* such that $\eta(\delta^*)$ is as large as possible in the direction. If $\eta(\delta^*) < \eta$ or $D(\delta^*) \neq D$, then reduce the size of δ^* until $\eta(\delta^*) > \eta$ and $D(\delta^*) = D$; Set $P_{k+1} = P(\delta^*)$ and $k := k + 1$, then go to Step 2.

If the algorithm does not stop in Step 4, then the directional derivative along Q is positive and therefore $\eta(\delta^*) > \eta$ if δ^* is small enough. In each iteration we keep the region unchanged and go along the direction with the largest performance derivative. This is the same as policy iteration in Markov decision processes (MDPs). Therefore, the above algorithm also possesses the similar properties of the policy iteration, such as the high convergence efficiency in most of the situations.

3.4 Some implementation issues

Because in each step the performance improves, the performance sequence η_k, $k = 1, 2, \cdots$, converges. However, following the deterministic sequence generated by (13), it may converge to a value less than the maximum. This may happen at the boundary points.

To avoid this, we need to add some randomness in Step 5 in the algorithm after Algorithm 1 stops. We modify it as

Step 5': Choose $1 > \theta > 0$ for all $i \in \{1, 2, \cdots, n\}$. With probability $1 - \theta$, set $Q = P' - P$. With probability θ, set $Q = P'' - P$, where P'' is any other matrix such that

$$\sum_{j \in D} (p''_{i,j} - p_{i,j})\rho_j > 0, \quad i = 1, 2, \cdots, n.$$

Set $P(\delta) = P + \delta Q$. □

The performance derivative along P to P'' is also positive; the iteration can be carried on. We call the algorithm with Step 5 replaced by Step 5' the *Modified Algorithm.*

Another implementation issue is the stopping criteria. In practice, the above algorithms may go on forever. This issue can be resolved in a standard way: choose a predetermined integer $K > 0$ and a small positive number $\epsilon > 0$, and stop the iteration process if in K consecutive iterations the improvement in performance is less than ϵ.

3.5 Properties of the algorithm

We say that a policy P reaches a local maximum if the directional derivative along any direction is non-positive. A policy P reaches a global maximum if no other point has a better performance.

It is reasonable to assume that every bank receives some payment, i.e., $l_i > 0, x_i > 0$ for all banks $i = 1, \cdots, n$. We can obtain the following Lemma 1.

Lemma 1. If P is not a boundary point and its performance η is a local maximum, it is also a global maximum.

This lemma is obvious based on PDF (11) and the directional derivative (12). For simplicity, we omit the proof.

Lemma 2. If Algorithm 1 stops in Step 4, it stops at a local maximum as well as a global maximum.

The proof of this lemma is omitted for the limit of space.

Both Algorithm 1 and the Modified Algorithm provide a sequence of P_k with increasing η_k. Since we take the Modified Algorithm after Algorithm 1 stops, the Modified Algorithm will present a better performance based on the result of Algorithm 1.

4. NUMERICAL EXAMPLES

In this section, we use some examples to demonstrate the effectiveness of our approach for reducing systemic risk.

Example 1. There are 3 banks in the system with

$$L = \begin{pmatrix} 0 & 60 & 40 \\ 20 & 0 & 60 \\ 10 & 30 & 0 \end{pmatrix},$$

$$\alpha = (50, 50, 100), \ b = (60, 80, 200).$$

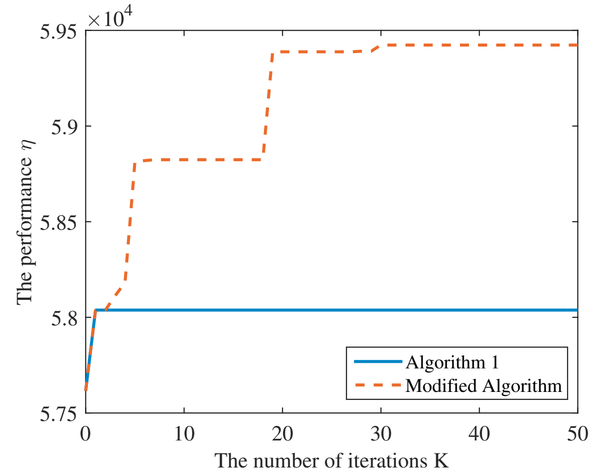

Fig. 1. The simulation result of Example 2

The pro rata payment matrix is

$$P = \begin{pmatrix} 0 & 0.375 & 0.250 \\ 0.125 & 0 & 0.375 \\ 0.042 & 0.125 & 0 \end{pmatrix}.$$

The original payment is $x = (68.185, 94.636, 152.535)$ by the Partition Algorithm. With this scheme, $D = \{1, 2, 3\}$, $N = \varnothing$, and a total payment $\eta = 315.356$.

We choose the pro rata scheme P as an initial policy and apply Algorithm 1. The algorithm stops at Step 4. As Lemma 2 states, Algorithm 1 converges to a local maximum as well as a global maximum. we get $x^* = (113.333, 113.333, 100.000)$ and $\eta^* = 366.667$. That means we reduce the system's loss by 20.97%. The new scheme is

$$P^* = \begin{pmatrix} 0 & 0.625 & 0 \\ 0.500 & 0 & 0 \\ 0.167 & 0 & 0 \end{pmatrix}.$$

We observe that Algorithm 1 only needs 1 iteration to converge in such an enormous policy space. Meanwhile, the optimal total payment is much larger than the original one. This demonstrates the efficiency of our Algorithm.

Example 2. There are 50 banks in the system with the liability matrix L and vectors α and b. For the limit of space we omit the detailed data of this example. Because of heavy liquidity, the whole system is confronted with a severe systemic crisis. With the pro rate scheme, 32 banks in the system will default, with a total payment $\eta = 57615.363$.

From the experiment result shown in Figure 1, we observe that Algorithm 1 converges near a boundary point with an effective improvement of performance. Algorithm 1 yields a new liquidation scheme P_1, with a total payment $\eta_1 = 58038.007$. This scheme curbs the financial contagion by reducing 2.69% of the system's loss. This is achieved only by improving the payment scheme, and no addition money is needed from the CB or government.

Then we apply the Modified Algorithm. We set the randomness parameter $\theta = 0.8$, then our algorithm yields an alternative liquidation scheme, with a total payment $\eta^* = 59423.384$ (see Figure 1). This is larger than both the results of Algorithm 1 and the pro rata scheme, reducing 11.51% of the system's loss. Therefore, the Modified Algorithm is demonstrated to be more efficient.

5. CONCLUSION

In this paper, we develop a sensitivity based view of financial systemic risk modeling to characterize analytically how the mechanism of default liquidation affects the total wealth of the financial system. We formulate the model as an optimization problem with equilibrium constraints and derive an optimal liquidation policy to minimize the system's loss. In contrast to most work on financial networks, in the case of default, we assume that all claimant nodes are paid off by an optimal liquidation policy decided by the central bank, instead of in proportion to the size of their claims on bank assets. During our derivation, we present a sensitivity based view and derive an algorithm to reduce the total loss of the financial system without changing the partition of default and non-default banks. Finally, we illustrate our model with examples.

We apply the direct-comparison based approach to solve the optimization problem and derive an efficient iteration algorithm to this highly nonlinear problem. Our work illustrates the advantages of the direct-comparison based approach, which was developed first for discrete event dynamic systems, and has been applied to solve many theoretical as well as practical problems. The central piece of the approach is the performance difference formula (PDF) that compares the performance under any two policies; we find that surprisingly the PDF for the financial risk contagion problem looks similar to that in MDPs. The research in this paper indicates that it can be applied to static problems as well.

Finally, this work focuses on one-region optimization of liquidation schemes. Next step, we can investigate optimization among multi regions and how to save banks from defaults. The higher order directional derivatives also deserve further investigations. Besides, we think that a dynamic version of the risk contagion model may measure the systemic risk better, and we hope to solve the dynamic problem in our further research.

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